PS2030

Political Research and Analysis

Unit 3: Models for Non-Continuous Dependent Variables

3. Models for Ordinal Outcomes

Spring 2025, Week 11

WW Posvar Hall 3600

Professor Steven Finkel

Models for Non-Binary, Non-Continuous Outcomes

- Many other kinds of non-continuous variables aside from the dichotomous or binary variables that we have considered so far with logit and probit models
  - Ordinal Outcomes: more than two ranked categories without necessarily equal distance between the categories
  - Multinomial Outcomes: more than two unranked categories
  - Count Outcomes: more than two non-negative integer categories
  - Censored Outcomes: continuous up to (or down to) a threshold
- We will have time only to discuss models for ordinal variables, but all of the other topics will be covered in Maximum Likelihood
- All involve extensions of either (or both) the non-linear specification or the latent variable framework for modeling dichotomous dependent variables via logit and probit regression

Modeling Ordinal Outcomes

- Ordinal variables have multiple categories that can be ranked
  - Social class: Low, Medium, High
  - 4-5 category “Strongly Agree” to “Strongly Disagree” survey questions
  - Outcomes of civil conflict: peace, low-level conflict, civil war
- Can you treat these variables as interval and estimate via OLS?
  - NO! OLS assumes equal distances between the categories, as in “every unit change in X brings about a β unit change in Y”. The units in Y must be equal, i.e., β at one point in the scale must be the same change as β at another point, and this is not necessarily the case with ordinal variables
  - Actually, the scale categories for ordinal variables are completely arbitrary anyway so the “unit change” idea is pretty meaningless. We could, e.g., assign a value of -400 to “low class”, 6,225 to “middle class”, and 4,500,823 to “high class”, or we could assign “1’ “2” “3”.
- So we estimate instead with “ordered probit” or “ordered logit” and ML methods

Ordered Probit

- Ordered Probit is a straightforward extension of the latent variable framework to take ordered categories into account. Instead of only one τ threshold for Y* at 0 to distinguish observations of “0” or “1”, we allow for multiple τ thresholds that distinguish observations of “category 1”, “category 2”, “category 3”, etc.
- As in dichotomous probit, model:

$$Y_i^* = \Sigma\beta X_i + \varepsilon_i$$

$$Y_i^* = X\mathrm{B} + \varepsilon_i \quad \text{in matrix type notation}$$

$$\mathrm{E}(Y_i^* \mid X) = X\mathrm{B}$$

- So Y* is continuous but unobserved. We map the observed variable Y_i to Y* via the “measurement equation” that says if Y* is above a certain threshold, observed Y will be 1; if Y* is above the next threshold, observed Y will be 2; above the next threshold, observed Y will be 3, and so on, depending on the number of categories

- Assume a three category ordinal variable
- Assign the zero threshold (τ_0) to be negative infinity (-∞) and the threshold for the last category (τ_3) to be positive infinity (∞)
- Then the full model is:

$$Y_i^* = \Sigma \beta X_i + \varepsilon_i$$

$$Y_i^* = X\mathrm{B} + \varepsilon_i \quad \text{in matrix type notation}$$

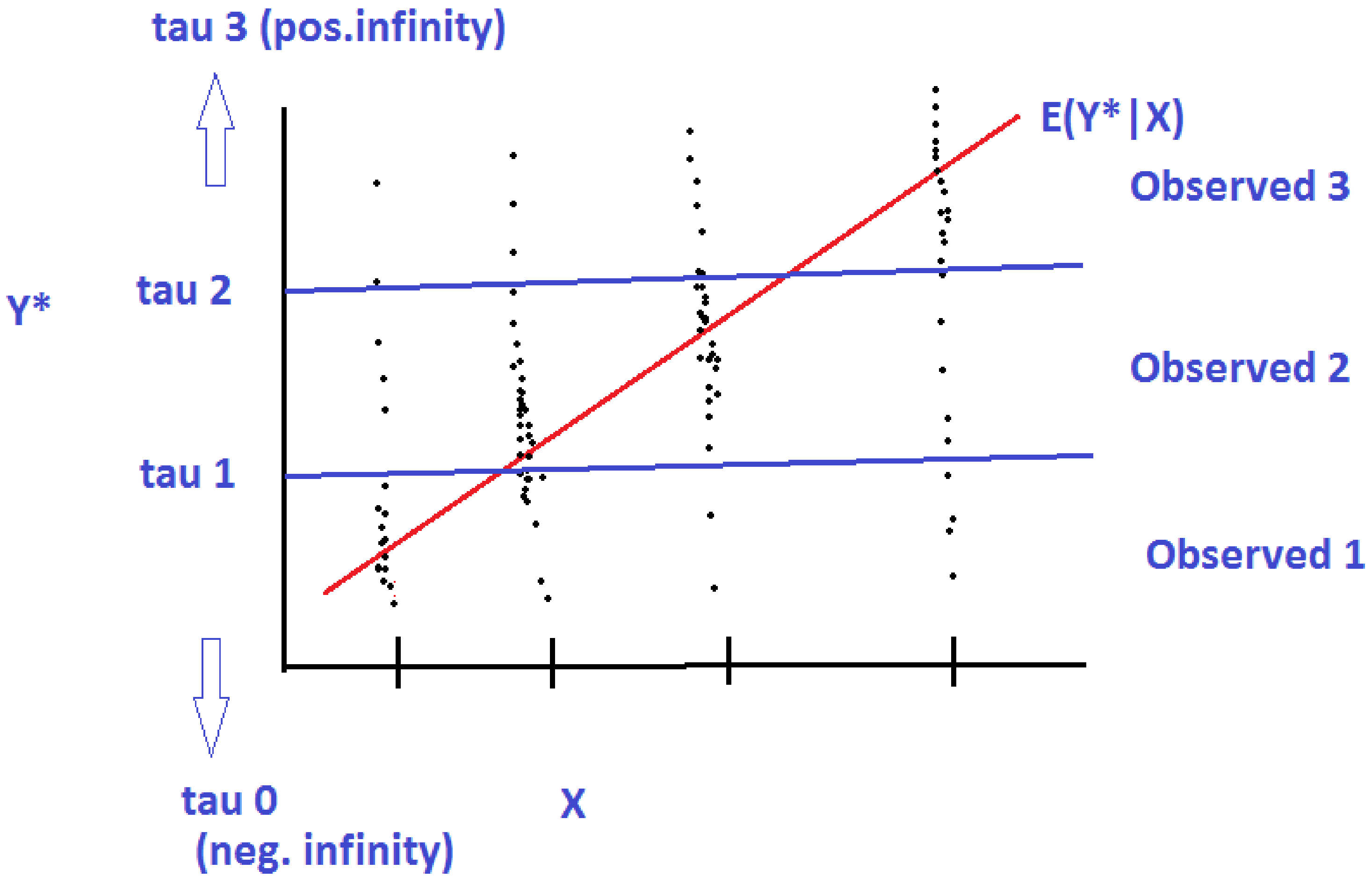
$$\mathrm{E}(Y_i^* \mid X) = X\mathrm{B}$$

with measurement equations

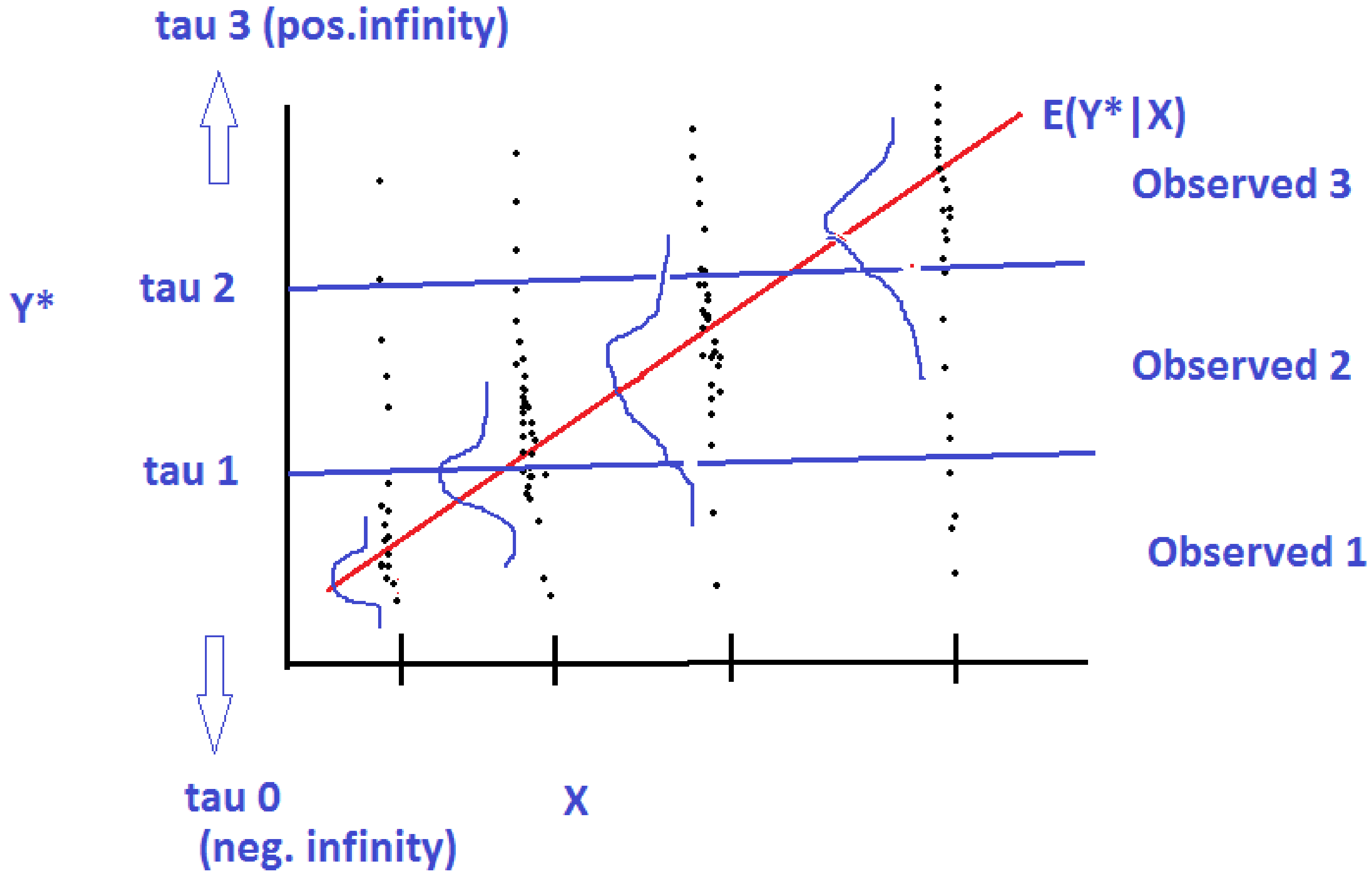
$$Y_i = 1 \text{ if } t_0(\text{-}\,\mu\,) <= \mathrm{Y}_i^* < t_1$$

$$Y_i = 2 \text{ if } t_1 <= \mathrm{Y}_i^* < t_2$$

$$Y_i = 3 \text{ if } t_2 <= \mathrm{Y}_i^* < t_3(\mu)$$


tau 3 (pos.infinity)
E(Y* | X)
Observed 3
Y*
tau 2
Observed 2
tau 1
Observed 1
tau 0
(neg. infinity)
X

- As with dichotomous probit, Y* is unobserved so cannot use OLS
- We also need to make assumptions about the error term ε. If we assume normality, we arrive at the "ordinal probit" specification. If we assume logistic distribution, we arrive at "ordinal logit" (though we will also arrive at ordinal logit by extending the non-linear probability framework a little later)
- Idea: XB takes Y* to some expected value, and then, depending on the size of the normally distributed error and whether it takes Y* past given thresholds, the observed Y will be 1, 2, or 3. We can use normal curve properties to calculate the probability, given XB, of obtaining an error term sufficiently large to put Y* over the τ_1 threshold, and over the τ_2 threshold, which would result in an observed Y of 2 or 3 respectively. Otherwise observed Y will be 1.



- Given XB, if ε is large enough, it will put even a very low Y* above the τ_1 or τ_2 thresholds; thus Y=2 or 3
- Given XB, if ε is small enough, it will put even a very high Y* below the τ_2 or τ_1 thresholds; thus Y=2 or 1

- The probability of observing, e.g., a "1" is:
- $P(Y = 1) \quad = P(\tau_0 \leq Y^* \leq \tau_1)$
- $P(Y = 1|X) = P(\tau_0 \leq XB + \varepsilon \leq \tau_1)$
- $P(Y = 1|X) = P(\tau_0 - XB \leq \varepsilon \leq \tau_1 - XB)$

- What is the probability that a normally distributed error term lies between two points? It is the difference in the cumulative probability associated with each of those points – like the distance between two z-scores.
- Intuitively: XB puts Y*, e.g., at 2. If τ_1 is, say, .5 , then we know that anytime there is an error term less than (.5-2=)**-1.5**, the person will be under the first threshold. Can there be a error term that will put the case under the τ_o? No, would have to be smaller than negative infinity! So we say that the probability of being in category 1 is the probability of the error term being less than (τ_1 – XB).

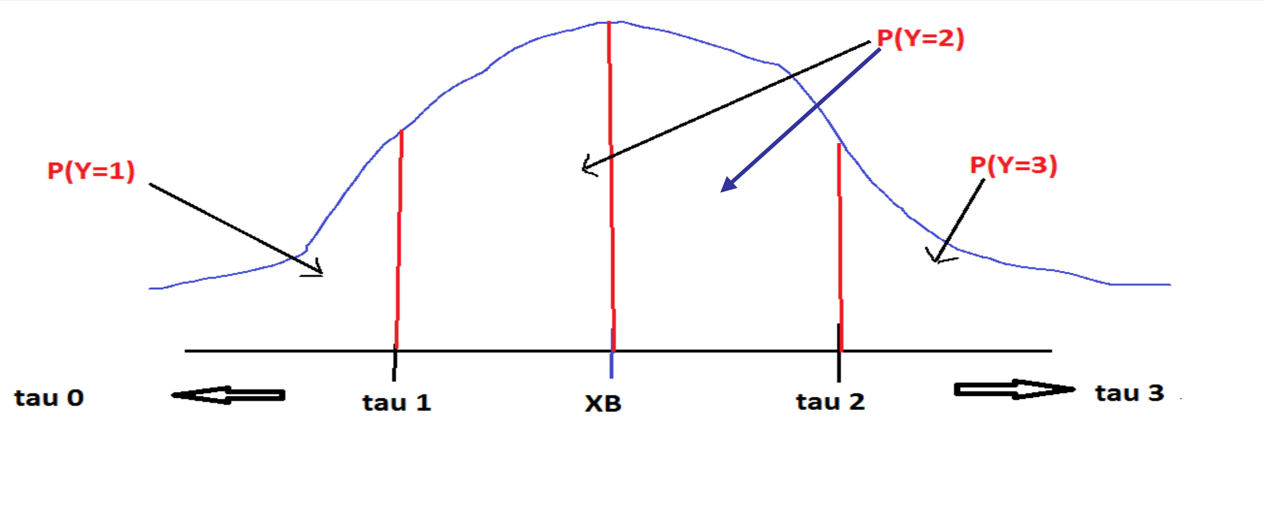
- Formally:

$$\mathrm{P(Y=1|X)}=\Phi(\tau_1 - X\mathrm{B}) - \Phi(\tau_0 - X\mathrm{B})$$

- The probability of observing a 1 is the proportion of the CDF (the cumulative normal distribution) associated with the first threshold minus XB, minus the CDF associated with the zero threshold minus XB. We know that the latter term must be 0, since the zero threshold is negative infinity and therefore has a CDF of 0.
- So:

$$\mathrm{P(Y=1|X)}=\Phi(\tau_1 - X\mathrm{B})$$

- The probability of observing a 1 is the proportion of the CDF associated with the first threshold minus XB. This gives the probability of obtaining an ε large enough to push Y* over the negative infinity threshold but not so large as to push Y* over τ_1.



Maps to: τ_1-XB 0 τ_2-XB

- Wherever XB takes Y*, an error term small enough will carry it below τ_1 and produce an observed Y or 1. This occurs with probability= $\Phi(\tau_1 - XB)$. If XB=4 and τ_1 equals 3, then error terms less than -1 will put Y* below the threshold, and Y=1. The probability of this occurring in any normal distribution is Φ (-1), or .16.
- Our example: XB=2, τ_1 =.5, then P(Y=1)= Φ (-1.5)=.067. So there is a 6.7% chance of observing Y=1 for a person with XB at 2 and τ_1 at .5. **P(Y=1|X)=.067**
- We can similarly work out the Ps associated with observing Y of 2 and 3

$$\mathrm{P}(Y=2\,|\,X)=\mathrm{P}(\tau_1 - X\mathrm{B}\,| \le \varepsilon < \tau_2 - X\mathrm{B})\,|\,X)$$

- which is the difference in the CDF associated with each of the points

$$\mathrm{P}(Y=2|\mathrm{X})=\Phi(\tau_2 - X\mathrm{B}) - \Phi(\tau_1 - X\mathrm{B})$$

- Take case of XB=2 again. If $\tau_2 = 3$, then the chance of being in category 2 is equal to the probability the error term is less than (3-2= 1) **and** greater than (.5-2= –1.5), which is the cut-point for getting into category 1. So Y=2 is whenever the error term is between –1.5 and 1.
- Verify this on the previous slide. We have the CDF associated with τ_2 in a normal distribution with XB as the mean, so it is the CDF associated with (τ_2 – XB) in the standard normal distribution. That gives P of being at or below the τ_2 threshold, or at or below 3. This is the *cumulative probability* of P(<=2). We then subtract from that the P of being in category 1, which is Φ(τ_1 – XB), to get P(Y=2) exactly.
- Φ(1)=.841 Φ(-1.5)=.067, so **P(Y=2|X)=.774**

$$\mathrm{P}(Y=3\,|\,X)=\mathrm{P}(\tau_2-X\mathrm{B}\,|\leq\varepsilon<\tau_3-X\mathrm{B})\,|\,X)$$

- Which is the probability of getting an error term large enough to put Y* over the τ_2 threshold but not so large to put Y* over the τ_3 threshold. Since τ_3 is positive infinity, it is impossible to get an error term larger, so we only really need to know whether the error term is greater than (τ_2 -XB). In terms of cumulative probabilities, the P(Y=3|X) would be the entire CDF minus the proportion of the CDF associated with P(Y<=2).

$$\mathrm{P}(Y=3\,|\,X)=\Phi(\tau_3-X\mathrm{B})-\Phi(\tau_2-X\mathrm{B})$$

$$\mathrm{P}(Y=3\,|\,X)=1-\Phi(\tau_2-X\mathrm{B})$$

- Our example: **P(Y=3|X)=1-.841 = .159**
- We can generalize all the probabilities as:

$$\mathrm{P}(Y=M\,|\,X)=\Phi(\tau_m-X\mathrm{B})-\Phi(\tau_{m-1}-X\mathrm{B})$$

ML Estimation of the Ordered Probit Model

- Given $\mathrm{P}(Y = M \mid X) = \Phi(\tau_m - X\mathrm{B}) - \Phi(\tau_{m-1} - X\mathrm{B})$
- we want to find the B, such that they maximize the joint probability of having observed the 1s, 2s, and 3s that we did observe in the sample

  $L = \Pi P_i$

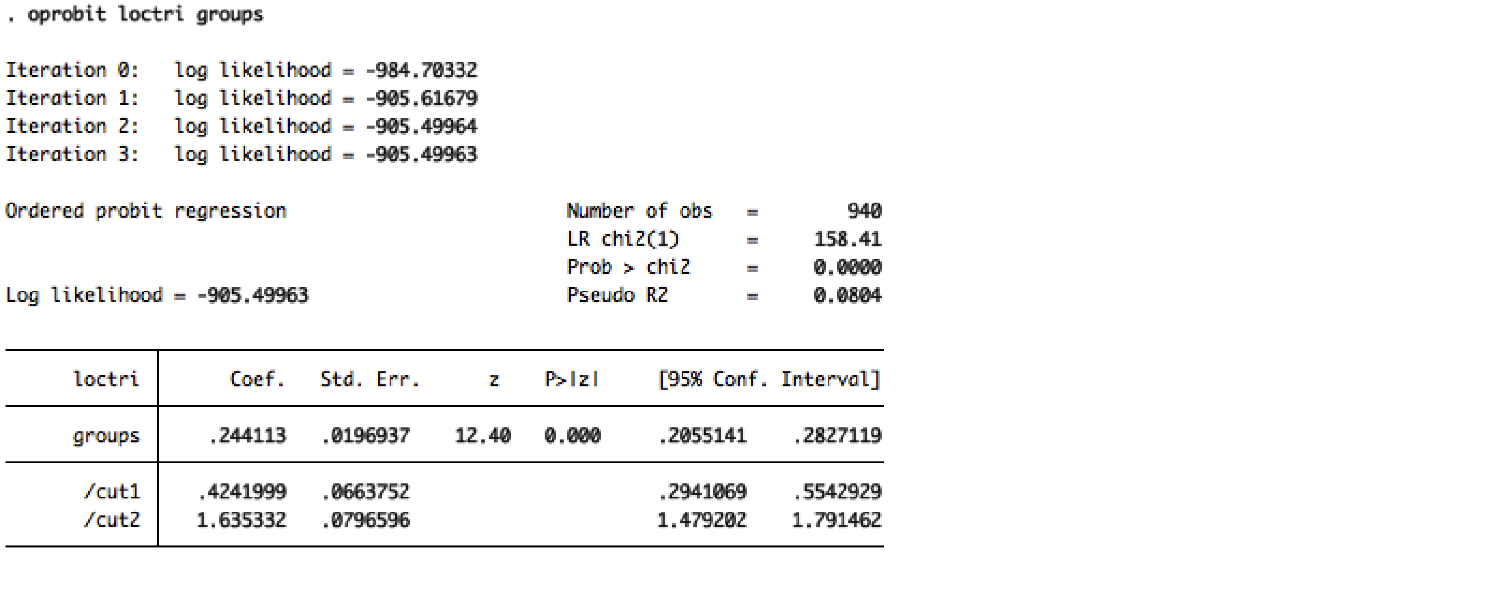
  $L = \Pi(\Phi(\tau_m - X\mathrm{B}) - \Phi(\tau_{m-1} - X\mathrm{B})),$

  for whatever category m that case i is in
- So if case 1 is in category 1, we use the P for M=1 in the likelihood, if case 2 is in category 3, we use the P for M=3, etc.
- We find the τ and the β that maximize:

  $\ln L = \Sigma \ln(\Phi(\tau_m - X\mathrm{B}) - \Phi(\tau_{m-1} - X\mathrm{B})$

  for the specific category j that individual i is in,

  summed over all the j categories



```
. oprobit loctri groups

Iteration 0:   log likelihood = -984.70332
Iteration 1:   log likelihood = -905.61679
Iteration 2:   log likelihood = -905.49964
Iteration 3:   log likelihood = -905.49963

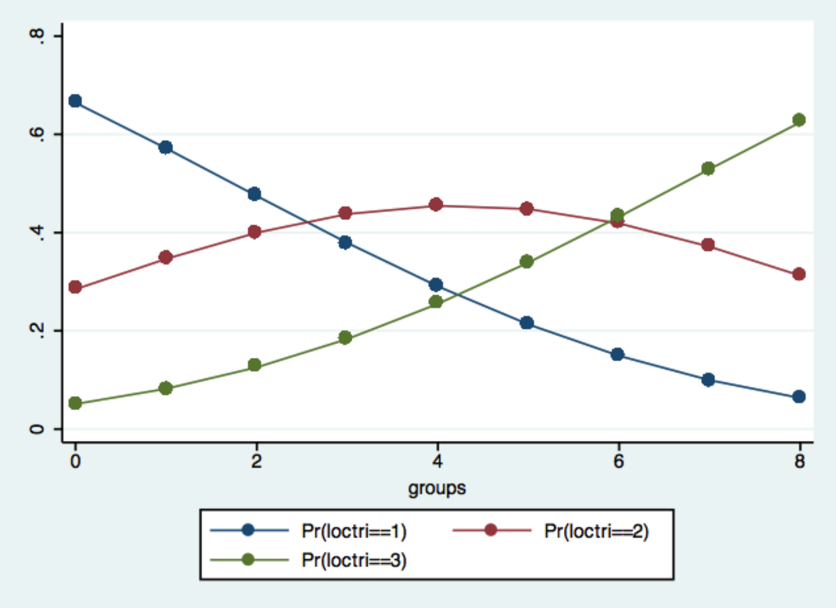
Ordered probit regression                         Number of obs   =        940
                                                  LR chi2(1)      =     158.41
                                                  Prob > chi2     =     0.0000
Log likelihood = -905.49963                       Pseudo R2       =     0.0804

------------------------------------------------------------------------------
      loctri |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      groups |    .244113   .0196937    12.40   0.000     .2055141    .2827119
-------------+----------------------------------------------------------------
       /cut1 |   .4241999   .0663752                      .2941069    .5542929
       /cut2 |   1.635332   .0796596                      1.479202    1.791462
------------------------------------------------------------------------------
```

- Can generate predicted P(Y=1), P(Y=2), P(Y=3) for all cases, depending on the level of X. These will yield non-linear P relationships with X for each outcome
- Display normal(.42-.244*4) is P(Y=1) is .29 for a person in 4 groups
- Display normal(1.65-.244*4)-normal(.42-.244*4) is P(Y=2) is .45 for a person in 4 groups
- Display 1- normal(1.65-.244*4) is P(Y=3) is .25 for a person in 4 groups

Additional Interpretations and Model Fit

- Individual significance tests based on LR or Wald
- Summary statistics for model and model comparisons
  - LR tests of nested models
  - McFadden's R-squared, McKelvey-Zavoina
  - Count and Adjusted Count R-squared
  - AIC and BIC entropy measures
- Use "MCHANGE" for changes in P(Y=M) as X changes by 1 unit, 1 standard unit, or marginal change
- Plot effects for better visualization
  - Effects of changes in variables on probability of being in categories 1/2/3 etc
  - Marginal effects of variables on all categories (mchangeplot in SPOST/STATA)
- Can calculate effects on Y* from either a unit change or a standard unit change in X using "listcoef"
  - Use standardized Y* given that variance of Y* is affected by inclusion or exclusion of sets of variables (as we discussed in context of binary outcomes)

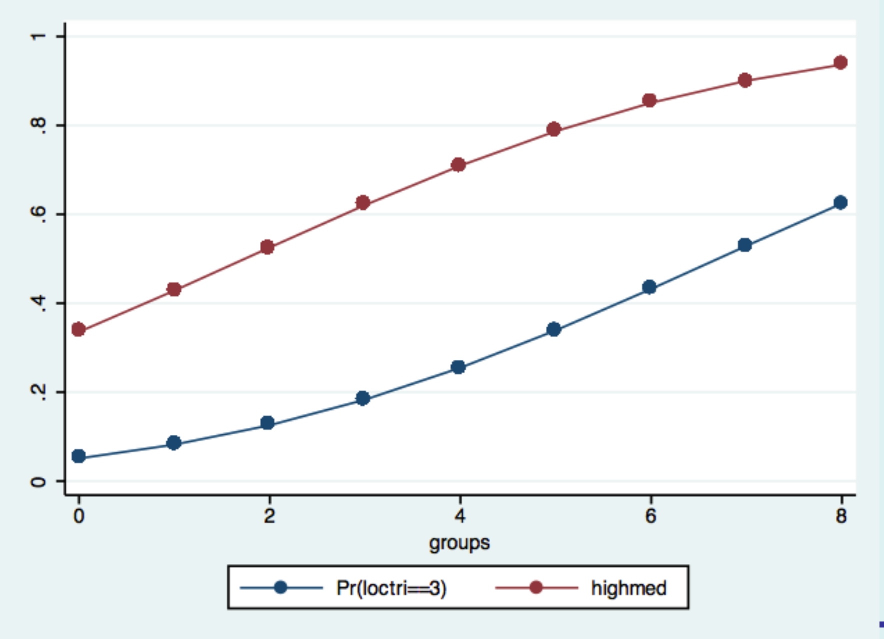


←

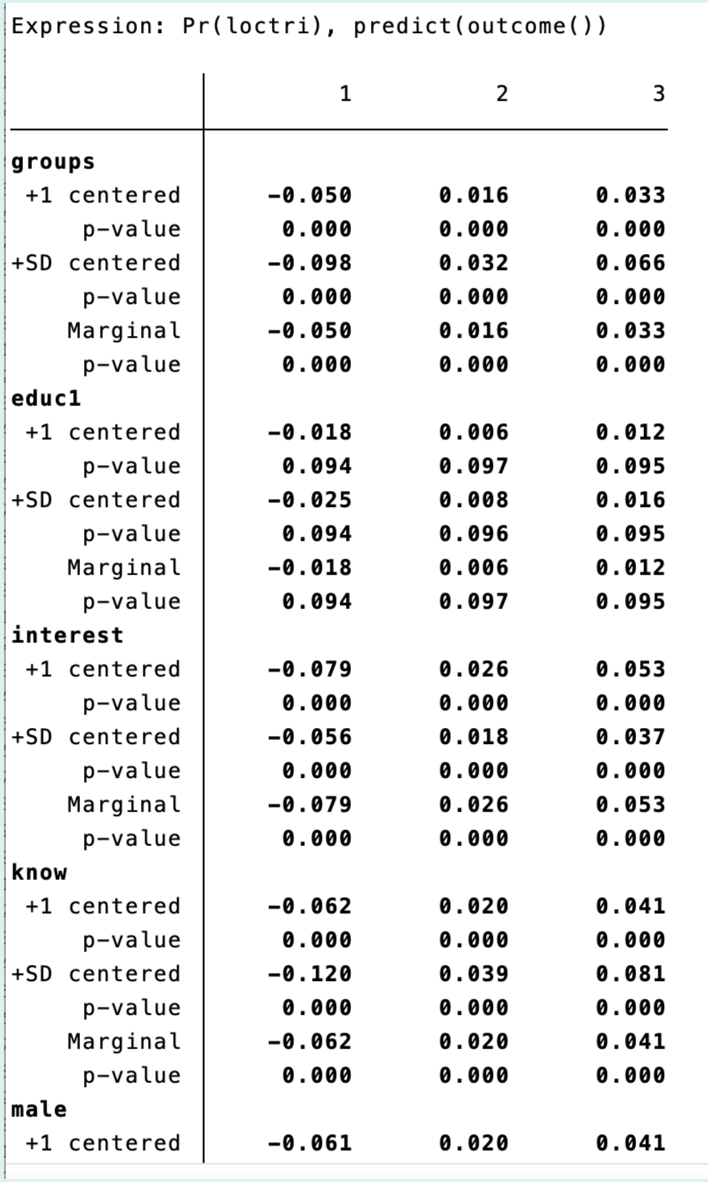
Predicted P for all categories at all levels of X

Predicted P of being in highest category, and highest or medium category, at all levels of X

→



Mchange and Mchangeplot for a Multivariate Model



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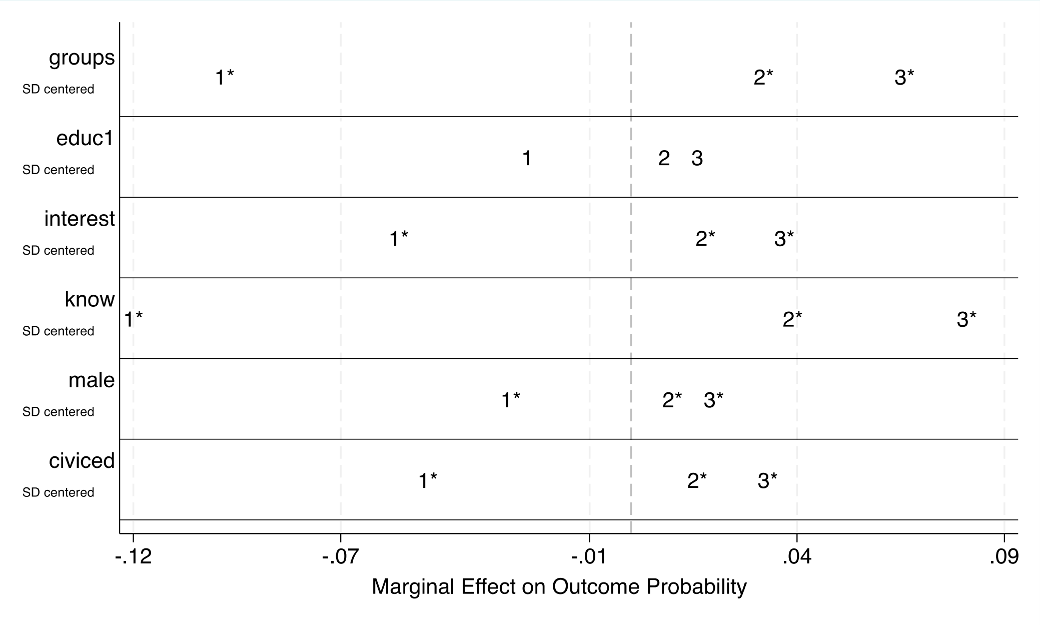
Expression: Pr(loctri), predict(outcome())

             |         1          2          3
-------------+---------------------------------
groups       |
 +1 centered |    -0.050      0.016      0.033
     p-value |     0.000      0.000      0.000
+SD centered |    -0.098      0.032      0.066
     p-value |     0.000      0.000      0.000
    Marginal |    -0.050      0.016      0.033
     p-value |     0.000      0.000      0.000
educ1        |
 +1 centered |    -0.018      0.006      0.012
     p-value |     0.094      0.097      0.095
+SD centered |    -0.025      0.008      0.016
     p-value |     0.094      0.096      0.095
    Marginal |    -0.018      0.006      0.012
     p-value |     0.094      0.097      0.095
interest     |
 +1 centered |    -0.079      0.026      0.053
     p-value |     0.000      0.000      0.000
+SD centered |    -0.056      0.018      0.037
     p-value |     0.000      0.000      0.000
    Marginal |    -0.079      0.026      0.053
     p-value |     0.000      0.000      0.000
know         |
 +1 centered |    -0.062      0.020      0.041
     p-value |     0.000      0.000      0.000
+SD centered |    -0.120      0.039      0.081
     p-value |     0.000      0.000      0.000
    Marginal |    -0.062      0.020      0.041
     p-value |     0.000      0.000      0.000
male         |
 +1 centered |    -0.061      0.020      0.041
```



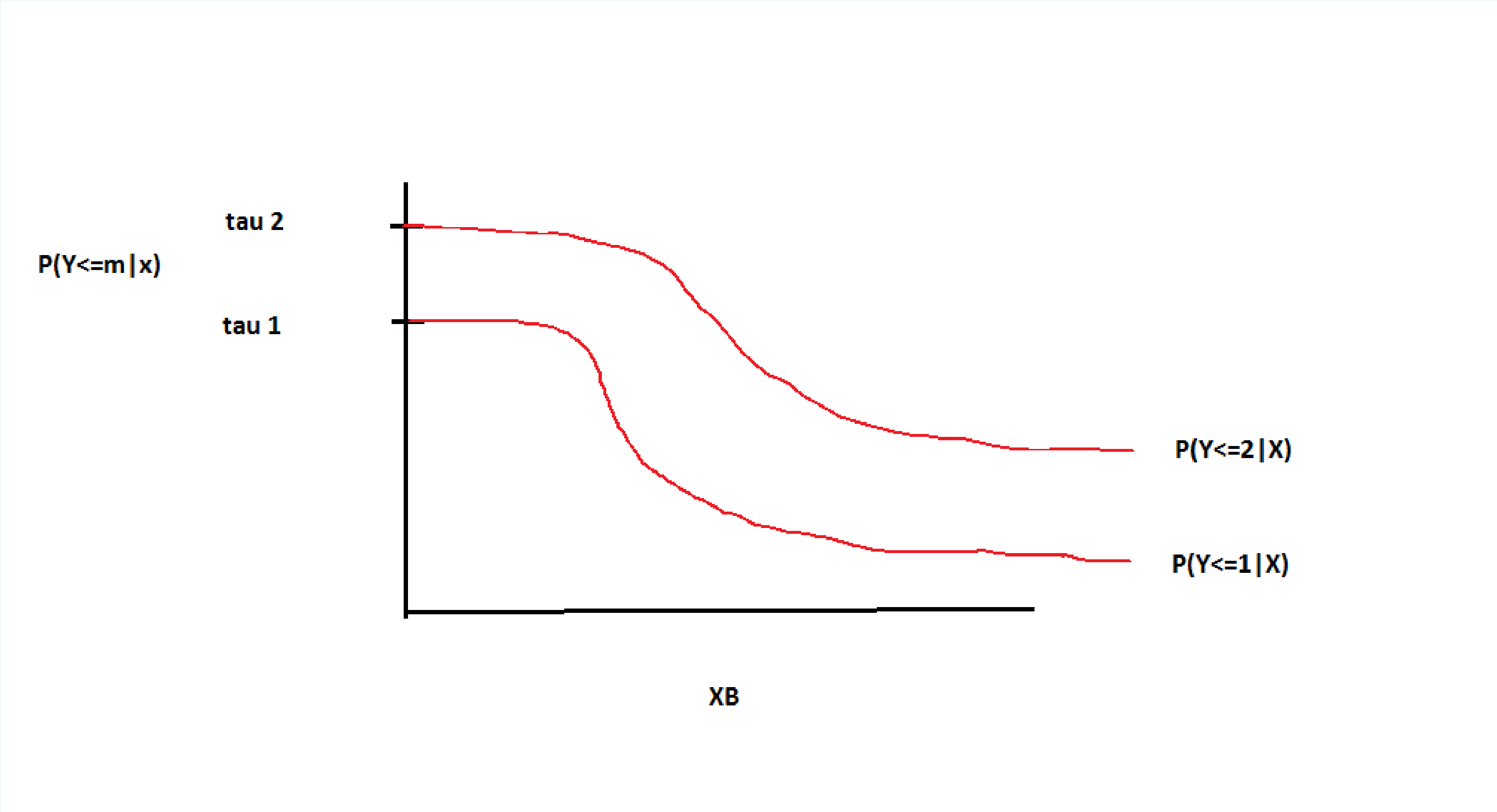
Note: Default change in mchangeplot is 1 SD change in the IV – can change this

Ordered Logit

- We can arrive at the same kind of probability model for ordered categorical variables by extending our earlier logit framework
- In the binary case we modeled the odds or the probability that a person/case is in category 1 versus category 0
- In the ordered case, we model the odds for the *cumulative probability* that a case is in category 1 or below, category 2 or below, etc.
- For a three category variable (low (1), Medium (2), High (3)):
- We say that the:
  - *cumulative probability* of being in category 1 is P(Y=1)
  - *cumulative probability* of being in category 2 is P(Y=1)+P(Y=2)
  - *cumulative probability* of being in category 3 is P(Y=1)+P(Y=2)+P(Y=3)=1
- If J is the number of categories, *j* are the individual categories j=1…J, and m is category m, then:

$$P(Y \leq m \mid X) = \sum P(Y = j \mid X) \text{ for j=1 to m}$$

- So if 3 categories, we get 2 cumulative probabilities P(Y<=2), P(Y<=1)
- If these cumulative probabilities are non-linear in relationship to X, bounded by 0 and 1 with Xs being unbounded, we can arrive at the same non-linear specification as in binary logit



- Why is curve sloped downward? We assume a positive relationship with X, that means that as X increases, the cumulative probability of being in category 1 **decreases**, and the cumulative probability of being in category 2 **decreases** as well (because Y will be higher)
- The ordered logit model:

$$\mathrm{P}(Y \leq m \mid X) = \frac{\exp^{\tau_m - X\mathrm{B}}}{1+\exp^{\tau_m - X\mathrm{B}}}$$

- **Having a negative XB here means that increases in X make the cumulative p smaller, i.e., a *positive* substantive relationship**
- NOTE: This would have looked the same had we done **regular dichotomous DV logit** by predicted P(Y=0) versus P(Y=1), instead of P(Y=1) versus P(Y=0), with τ=0

$$\mathrm{P}(Y = 0 \mid X) = \frac{\exp^{-X\mathrm{B}}}{1+\exp^{-X\mathrm{B}}} = \frac{1}{1+\exp^{X\mathrm{B}}}$$

- In terms of odds:

Cumulative Odds of being in category less than or equal to m, compared to being in category greater than m:

$$\frac{P(Y \leq m \mid x)}{P(Y > m \mid x)} = \exp^{\tau_m - X\mathrm{B}}$$

- Taking logs of both sides gives the log-cumulative odds as:

$$\ln \frac{P(Y \leq m \mid x)}{P(Y > m \mid x)} = \tau_m - X\mathrm{B}$$

- So ordered logit is linear in the **cumulative logits**, or "log-cumulative odds" that Y is in category m or lower
- Increases in X lead the cumulative logit to **decrease** by B amount, which means that the odds are smaller that the case is in lower categories as X gets larger (if B is positive, that is)

- We can derive probabilities of being in each category

$$P(Y = 1 | X) = \frac{\exp^{\tau_1 - X\mathrm{B}}}{1 + \exp^{\tau_1 - X\mathrm{B}}}$$ which is the same as the cumulative P(Y<=1)

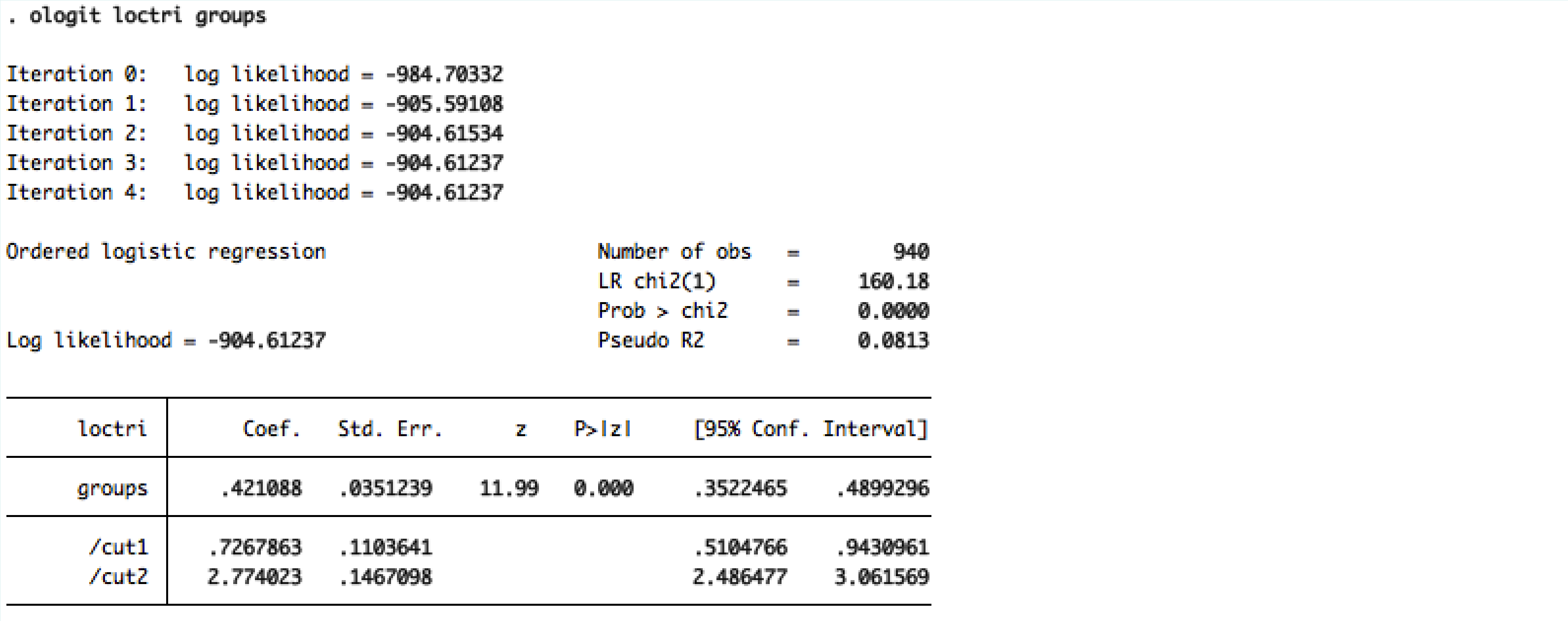
$$P(Y = 2 | X) = \frac{\exp^{\tau_2 - X\mathrm{B}}}{1 + \exp^{\tau_2 - X\mathrm{B}}} - \frac{\exp^{\tau_1 - X\mathrm{B}}}{1 + \exp^{\tau_1 - X\mathrm{B}}}$$

or the cumulative probability of P(Y<=2) minus the cumulative P(Y<=1)

$$P(Y = 3 | X) = 1 - \frac{\exp^{\tau_2 - X\mathrm{B}}}{1 + \exp^{\tau_2 - X\mathrm{B}}}$$

which is 1 minus the cumulative probability of P(Y<=2)

Example of Ordered Logit



```
. ologit loctri groups

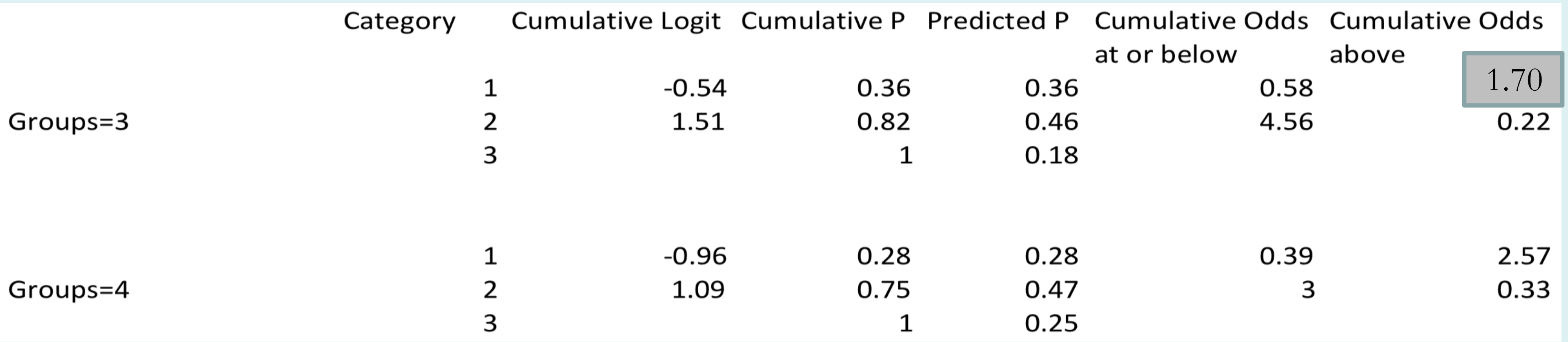
Iteration 0:   log likelihood = -984.70332
Iteration 1:   log likelihood = -905.59108
Iteration 2:   log likelihood = -904.61534
Iteration 3:   log likelihood = -904.61237
Iteration 4:   log likelihood = -904.61237

Ordered logistic regression                       Number of obs   =        940
                                                  LR chi2(1)      =     160.18
                                                  Prob > chi2     =     0.0000
Log likelihood = -904.61237                       Pseudo R2       =     0.0813

------------------------------------------------------------------------------
      loctri |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      groups |    .421088   .0351239    11.99   0.000     .3522465    .4899296
-------------+----------------------------------------------------------------
       /cut1 |   .7267863   .1103641                      .5104766    .9430961
       /cut2 |   2.774023   .1467098                      2.486477    3.061569
------------------------------------------------------------------------------
```

- So cumulative logit for category 1: .72-.42*groups
- Cumulative logit for category 2: 2.77-.42*groups
- Predicted Probability for category 1: exp(.72-.42*groups)/(1+exp(.72-.42*groups))
- Predicted Probability for category 2:

  exp(2.77-.42*groups)/(1+ exp(2.77-.42*groups)) - exp(.72-.42*groups)/(1+exp(.72-.42*groups)
- Predicted Probability for category 3: 1- exp(2.77-.42*groups)/(1+ exp(2.77-.42*groups))



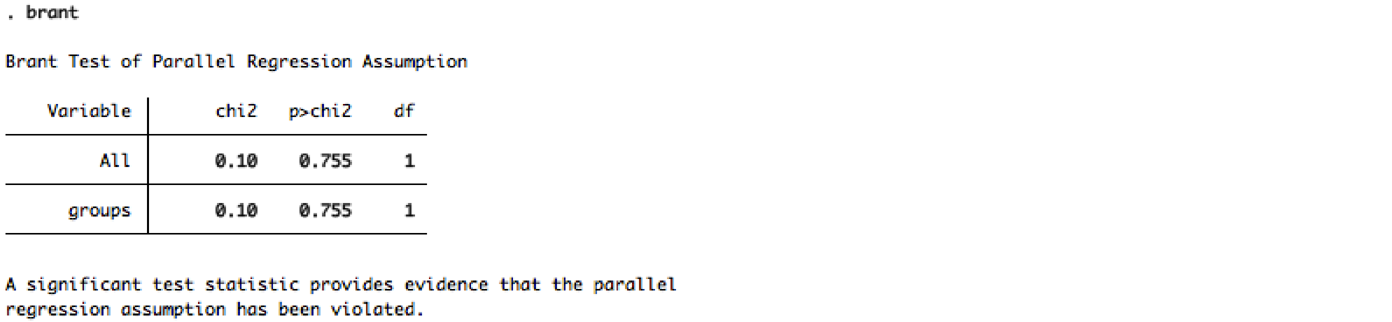
	Category	Cumulative Logit	Cumulative P	Predicted P	Cumulative Odds at or below	Cumulative Odds above
	1	-0.54	0.36	0.36	0.58	1.70
Groups=3	2	1.51	0.82	0.46	4.56	0.22
	3		1	0.18		
	1	-0.96	0.28	0.28	0.39	2.57
Groups=4	2	1.09	0.75	0.47	3	0.33
	3		1	0.25		

- Odds Interpretation:
- As groups increase by one unit, the cumulative odds of being in category *m or below* versus being *above category m* changes by a factor of exp(-β).
- Here it is exp(-.42)=.66
- So cumulative odds of being at or below category 1 versus above for a groups=4 person compared to a groups=3 person is .39/.58=.67
- Cumulative odds of being at or below category 2 or below versus above for a groups=4 person compared to a groups=3 person is 3/4.56=.66
- Alternative (and easier) interpretation is to express in terms of the **greater** odds of being above category *m* versus in a category *at or smaller than category m* as exp(β), or 1.52 in this case
- This is what is given by "listcoef" in STATA, along with other standardized effects

The Parallel Regression Assumption

- Important assumption of Ordered Logit or Probit: There is only one β for each X – that is, the lines are *parallel* for the cumulative Ps for all categories. This means that the effect of X on getting into category 1 is the threshold for category 1 – XB (or the change in the cumulative log-odds for category 1), the effect of getting into category 2 is the threshold for category 2-XB (or the change in the cumulative log-odds for category 2), etc., **and the β are the same for all of these calculations**. It is *not* the case that in predicting P(Y=2) that you use a different value of β than you use to predict P(Y=3), P(Y=1), etc.
- So changing from groups =3 to groups=4 leads to
  - A factor change in the cumulative odds of being at/below versus being above category 1 of .66, or .39/.58 (or a 1.52 factor change in the odds of being greater than category 1 versus at or below: 2.57/1.70=1.51)
  - A factor change in the cumulative odds of being at/below versus being above category 2 of .66, or 3.00/4.56 (or a 1.52 factor change in the odds of being greater than category 2 versus at or below .33/.22=1.5)
- The change in cumulative odds at or below versus above is a constant .66 **NO MATTER WHICH CATEGORY YOU ARE TALKING ABOUT**

- Could imagine a situation, though, where cumulative P (or odds) of being at or below category 1, e.g., would be affected by X to one degree, and the cumulative P of being at or below category 2, e.g., would be affected by X to a different degree
- Could run a bunch of different bivariate logits based on the different cumulative P values, and compare the coefficients for X to see if they are (nearly) identical
- "Brant" Test available in Stata: brant,detail



```
. brant

Brant Test of Parallel Regression Assumption

    Variable |      chi2   p>chi2    df
-------------+--------------------------
         All |      0.10    0.755     1
-------------+--------------------------
      groups |      0.10    0.755     1
----------------------------------------

A significant test statistic provides evidence that the parallel
regression assumption has been violated.
```

- If significant, proportional regression assumption is violated, and you need to move to alternatives (multinomial logit, "partial proportional odds", or others) which we won't have time for this semester!